**Exercise (1)**

(1) Represent the following function using unit step functions. Find its Laplace transform

(a) $t \quad (0 < t < 1)$

(b) $\sin 3t \quad (0 < t < \pi)$

(c) $\cos \pi \, \mathrm{t} \; (1 < \mathrm{t} < 4)$

(2) Find inverse Laplace transform for the following function

(a) $F(s) = \dfrac{se^{-s}}{s^2 + \omega^2}$ (b) $F(s) = \dfrac{e^{-4s}}{s^2}$

(c) $F(s) = \left(e^{-2\pi s} - e^{-8\pi s}\right) / \left(s^2 + 1\right)$ (d) $F(s) = \left(\dfrac{e^{-2\pi s} - e^{-8\pi s}}{s}\right)$

(3) Using Laplace Transform solve the following differential Equation

(a) $y'' + 2y' + 2y = 0, \qquad y(0) = 0, \; y'(0) = 1$

(b) $9y'' - 6y' + y = 0, \qquad y(0) = 3, \; y'(0) = 1$

**<u>Answer Question (1)</u>**

(a) $f(t) = t\left[1 - u(t-1)\right] = t - tu(t-1) = \dfrac{1}{s} - e^{-s} L\{t+1\} = \dfrac{1}{s} - e^{-s}\left(\dfrac{1}{s^2} + \dfrac{1}{s}\right)$

(b) $f(t) = \sin 3t\left[1 - u(t-\pi)\right] = \dfrac{3}{s^2+9} - e^{-\pi s} L\{\sin 3(t+\pi)\}$

$$= \frac{3}{s^2+9} - e^{-\pi s} L\{-\sin 3t\} = \frac{3}{s^2+9} + \frac{3}{s^2+9} e^{-\pi s} = 3\left(\frac{1+e^{-\pi s}}{s^2+9}\right)$$

(c) $f(t) = \cos \pi t\left[u(t-1) - u(t-4)\right] = \cos \pi t \, u(t-1) - \cos \pi t \, u(t-4)$

$$= e^{-s} L\{\cos \pi(t+1)\} - e^{-4t} L\{\cos \pi(t+4)\} = e^{-s} L\{-\cos \pi t\} - e^{-4t} L\{\cos \pi t\}$$

$$= -e^{-s} \frac{s}{s^2+\pi^2} - e^{-4s} \frac{s}{s^2+\pi^2} = \frac{-s\left(e^{-s} + e^{-4s}\right)}{s^2+\pi^2}$$

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**Answer Question (2)**

(a) $F(s)=\dfrac{se^{-s}}{s^2+\omega^2}$ $\quad\therefore f(t)=u(t-1)L^{-1}\left\{\dfrac{s}{s^2+\omega^2}\right\}_{t\to t-1}=u(t-1).\cos\omega(t-1)$

(b) $F(s)=\dfrac{e^{-4s}}{s^2}$ $\quad\therefore f(t)=u(t-4)L^{-1}\left\{\dfrac{1}{s^2}\right\}_{t\to t-4}=u(t-4).(t-4)$

(c) $F(s)=\left(e^{-2\pi s}-e^{-8\pi s}\right)/\left(s^2+1\right)$

$$\therefore f(t)=u(t-2\pi)L^{-1}\left\{\frac{1}{s^2+1}\right\}_{t\to t-2\pi}-u(t-8\pi)L^{-1}\left\{\frac{1}{s^2+1}\right\}_{t\to t-8\pi}$$

$$=u(t-2\pi)\sin(t-2\pi)-u(t-8\pi)\sin(t-8\pi)=u(t-2\pi)\sin t-u(t-8\pi)\sin t$$

$$=\sin t\left[u(t-2\pi)-u(t-8\pi)\right]$$

(d) $F(s)=\left(\dfrac{e^{-2\pi s}-e^{-8\pi s}}{s}\right)$

$$\therefore f(t)=u(t-2\pi)L^{-1}\left\{\frac{1}{s}\right\}_{t\to t-2\pi}-u(t-8\pi)L^{-1}\left\{\frac{1}{s}\right\}_{t\to t-8\pi}$$

$$=u(t-2\pi)-u(t-8\pi)$$

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**Answer Question (3)**

(a) $y''+2y'+2y=0,\qquad y(0)=0,\ y'(0)=1$

**Answer**

$L\left\{y''+2y'+2y\right\}=0$

$s^2Y-sy(0)-y'(0)+2\left(sY-y(0)\right)+2Y=0$

$s^2Y-\cancel{sy(0)}-1+2\left(sY-\cancel{y(0)}\right)+2Y=0$

$Y=\dfrac{1}{s^2+2s+2}$ and $y(t)=L^{-1}\left\{\dfrac{1}{s^2+2s+1+1}\right\}=L^{-1}\left\{\dfrac{1}{(s+1)^2+1}\right\}=e^{-t}\sin t$

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(b) $9y''-6y'+y=0,\qquad y(0)=3,\ y'(0)=1$

$$L\{9y'' - 6y' + y\} = 0$$

$$9s^2 Y - 9sy(0) - 9y'(0) - 6sY + 6y(0) + Y = 0$$

$$9s^2 Y - 27s - 9 - 6sY + 18 + Y = 0$$

$$Y = \frac{27s - 9}{9s^2 - 6s + 1} = \frac{27s - 9}{(3s - 1)^2} = \frac{9}{(3s - 1)}$$

$$y(t) = L^{-1}\left\{\frac{9}{(3s - 1)}\right\} = L^{-1}\left\{\frac{3}{(s - 1/2)}\right\} = 3e^{\frac{1}{3}t}$$

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**Exersice (2)**

(1) Find Laplace transform for the following

(a) $f(t) = t\cosh\omega t$ (b) $f(t) = t\sinh\omega t$

(c) $f(t) = t^2\cosh\omega t$ (d) $f(t) = t^2\sinh\omega t$

**<u>Answer Question (1)</u>**

(a) $$L\{t\cosh\omega t\} = -\frac{d}{ds}L\{\cosh\omega t\} = -\frac{d}{ds}\frac{s}{s^2 - \omega^2} = \frac{\omega^2 + s^2}{\left(s^2 - \omega^2\right)^2}$$

$$L\{t\cosh\omega t\} = \tfrac{1}{2}L\left\{t\left(e^{\omega t} + e^{-\omega t}\right)\right\} = \tfrac{1}{2}L\left\{te^{\omega t}\right\} + \tfrac{1}{2}L\left\{te^{-\omega t}\right\}$$

$$= \tfrac{1}{2}L\{t\}_{s \to s-\omega} + \tfrac{1}{2}L\{t\}_{s \to s+\omega} = \tfrac{1}{2}\left(\frac{1}{(s-\omega)^2} + \frac{1}{(s+\omega)^2}\right) = \frac{\omega^2 + s^2}{\left(s^2 - \omega^2\right)^2}$$

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(b) $$L\{t\sinh\omega t\} = -\frac{d}{ds}L\{\sinh\omega t\} = -\frac{d}{ds}\frac{\omega}{s^2 - \omega^2} = \frac{2\omega s}{\left(s^2 - \omega^2\right)^2}$$

$$L\{t\sinh\omega t\} = \tfrac{1}{2}L\left\{t\left(e^{\omega t} - e^{-\omega t}\right)\right\} = \tfrac{1}{2}L\left\{te^{\omega t}\right\} - \tfrac{1}{2}L\left\{te^{-\omega t}\right\}$$

$$=\tfrac{1}{2}L\{t\}_{s\to s-\omega}-\tfrac{1}{2}L\{t\}_{s\to s+\omega}=\tfrac{1}{2}\left(\frac{1}{(s-\omega)^2}-\frac{1}{(s+\omega)^2}\right)=\frac{2\omega s}{\left(s^2-\omega^2\right)^2}$$

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(c) $L\left\{t^2\cosh\omega t\right\}=\dfrac{d^2}{ds^2}L\{\cosh\omega t\}=\dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2-\omega^2}\right)=\dfrac{2s^3+6s\omega^2}{\left(s^2-\omega^2\right)^3}$

$$L\left\{t^2\cosh\omega t\right\}=\tfrac{1}{2}L\left\{t^2\left(e^{\omega t}+e^{-\omega t}\right)\right\}=\tfrac{1}{2}L\left\{t^2e^{\omega t}\right\}+\tfrac{1}{2}L\left\{t^2e^{-\omega t}\right\}$$

$$=\tfrac{1}{2}L\left\{t^2\right\}_{s\to s-\omega}+\tfrac{1}{2}L\left\{t^2\right\}_{s\to s+\omega}$$

$$=\left(\frac{1}{(s-\omega)^3}+\frac{1}{(s+\omega)^3}\right)=\frac{2s^3+6s\omega^2}{\left(s^2-\omega^2\right)^3}$$

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(c) $L\left\{t^2\sinh\omega t\right\}=\dfrac{d^2}{ds^2}L\{\sinh\omega t\}=\dfrac{d^2}{ds^2}\left(\dfrac{\omega}{s^2-\omega^2}\right)=\dfrac{2\omega^3+6s\omega^2}{\left(s^2-\omega^2\right)^3}$

$$L\left\{t^2\sinh\omega t\right\}=\tfrac{1}{2}L\left\{t^2\left(e^{\omega t}-e^{-\omega t}\right)\right\}=\tfrac{1}{2}L\left\{t^2e^{\omega t}\right\}-\tfrac{1}{2}L\left\{t^2e^{-\omega t}\right\}$$

$$=\tfrac{1}{2}L\left\{t^2\right\}_{s\to s-\omega}-\tfrac{1}{2}L\left\{t^2\right\}_{s\to s+\omega}$$

$$=\left(\frac{1}{(s-\omega)^3}-\frac{1}{(s+\omega)^3}\right)=\frac{2\omega^3+6s\omega^2}{\left(s^2-\omega^2\right)^3}$$

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**<u>Answer Question (2)</u>**

(2) Find the Laplace transform of the periodic function $f(t)$.

(a) $f(t)=\begin{cases}1 & 0<t<a\\ 0 & a<t<2a\end{cases}$ period $2a$ $\qquad Ans.\ \dfrac{1}{s(1+e^{-as})}$

let $f_1(t)=u(t)-u(t-a)$  $L\{f_1(t)\}=\dfrac{1-e^{-as}}{s}$

$$\therefore L\{f(t)\}=\frac{1}{1-e^{-2as}}\left(\frac{1-e^{-as}}{s}\right)=\frac{1}{(1+e^{-as})(1-e^{-as})}\left(\frac{1-e^{-as}}{s}\right)=\frac{1}{s(1+e^{-as})}$$

---

(b) $f(t)=f(t)=\begin{cases}1 & 0<t<a\\ -1 & a<t<2a\end{cases}$ period $2a$  $Ans.\dfrac{(1-e^{-as})}{s(1+e^{-as})}=\dfrac{1}{s}\tanh\dfrac{as}{2}$

let $f_1(t)=u(t)-2u(t-a)+u(t-2a)$ then $L\{f_1(t)\}=\dfrac{1-2e^{-as}+e^{-2as}}{s}=\dfrac{\left(1-e^{-as}\right)^2}{s}$

$$\therefore L\{f(t)\}=\frac{1}{1-e^{-2as}}\frac{\left(1-e^{-as}\right)^2}{s}=\frac{1}{(1+e^{-as})(1-e^{-as})}\frac{\left(1-e^{-as}\right)^2}{s}=\frac{\left(1-e^{-as}\right)}{s\left(1+e^{-as}\right)}$$

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(c) $f(t)=f(t)=\begin{cases}t & 0<t<a\\ 0 & a<t<2a\end{cases}$ period $2a$

let $f_1(t)=t\left[\,u(t)-u(t-a)\right]$

$$L\{f_1(t)\}=\frac{1}{s^2}-e^{-as}L\{t+a\}=\frac{1}{s^2}-e^{-as}\left(\frac{1}{s^2}+\frac{a}{s}\right)=\frac{1-e^{-as}-ase^{-as}}{s^2}$$

$$\therefore L\{f(t)\}=\frac{1}{1-e^{-2as}}\left(\frac{1-e^{-as}-ase^{-as}}{s^2}\right)=\frac{1-e^{-as}}{s^2(1+e^{-as})(1-e^{-as})}-\frac{1}{1-e^{-2as}}\frac{ase^{-as}}{s^2}$$

$$=\frac{1}{\mathrm{s}^2(1+e^{-as})}-\frac{ae^{-as}}{s(1-e^{-2as})}$$

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(d) $f(t)=f(t)=\begin{cases}t & 0<t<a\\ 2a-t & a<t<2a\end{cases}$ period $2a$

let $f_1(t)=tu(t)+(2a-2t)u(t-a)-(2a-t)u(t-2a)$

$$= tu(t) - 2(t-a)u(t-a) + (t-2a)u(t-2a)$$

$$L\{f_1(t)\} = \frac{1}{s^2} - 2e^{-as}L\{t\} + e^{-2as}L\{t\} = \frac{1}{s^2}\left(1 - 2e^{-as} + e^{-2as}\right) = \frac{1}{s^2}\left(1 - e^{-as}\right)^2$$

$$\therefore L\{f(t)\} = \left(\frac{1}{1-e^{-2as}}\right) . \frac{1}{s^2}\left(1-e^{-as}\right)^2 = \frac{1-e^{-as}}{s^2(1+e^{-as})} = \frac{1}{s^2}\tanh\frac{as}{2}$$

**<u>Answer Question (3)</u>**

(3) Compute the Laplace transform of the function

$f(t) = u(t) - u(t-a) + u(t-2a) - u(t-3a) + \cdots$ term by term and compare with **Question 2(a).**

**Answer**

$$f(t) = u(t) - u(t-a) + u(t-2a) - u(t-3a) + \cdots$$

$$L\{f(t)\} = L\{u(t) - u(t-a) + u(t-2a) - u(t-3a) + \cdots\}$$

$$= \frac{1}{s}\left(1 - e^{-as} + e^{-2as} - e^{-3as} + .....\right) \quad \text{and} \quad L\{f(t)\} = \frac{1}{s}\left(\frac{1}{1+e^{-as}}\right)$$

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**<u>Answer Question (4)</u>**

(4) Express the function in Question 1(b) as an infinite series of unit step functions and compute its Laplace transform term by term.

**Answer**

$$f(t) = \begin{cases} 1 & 0 < t < a \\ -1 & a < t < 2a \end{cases} \quad \text{period } 2a$$

$$f(t) = u(t) - 2u(t-a) + 2u(t-2a) - 2u(t-3) + \ldots$$

$$\mathrm{L}\{f(t)\} = \frac{1}{s}\left(1 - 2e^{-as} + 2e^{-2as} - 2e^{-3as} + \ldots\right) = \frac{1}{s}\left(-1 + 2 - 2e^{-as} + 2e^{-2as} - 2e^{-3as} + \ldots\right)$$

$$\mathrm{L}\{f(t)\} = \frac{1}{s}\left(-1 + 2\left\{1 - e^{-as} + e^{-2as} - e^{-3as} + \ldots\right\}\right) = \frac{1}{s}\left(\frac{2}{1+e^{-as}} - 1\right)$$

$$= \frac{1}{s}\left(\frac{1+e^{-as}}{1+e^{-as}}\right) = \frac{1}{s}\tanh\frac{as}{2}$$

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**Answer Question (5)**

(5) Determine $f(t)=L^{-1}\{F(s)\}$ where $F(s)=\dfrac{1-e^{-as}}{s(e^{as}+e^{-as})}$ by writing $F(s)$ as an infinite series of exponential functions and computing the inverse term by term.

$$F(s)=\frac{1-e^{-as}}{s(e^{as}+e^{-as})}=\frac{1}{se^{as}}\frac{1-e^{-as}}{\left(1+e^{-2as}\right)}=\frac{1}{s}\left(e^{-as}-e^{-2as}\right)\left(1+e^{-2as}\right)^{-1}$$

$$=\frac{1}{s}\left(e^{-as}-e^{-2as}\right)\left(1-e^{-2as}+e^{-4as}-e^{-6as}+e^{-8as}-e^{-10as}+\ldots\right)$$

$$=\frac{1}{s}\left(e^{-as}-e^{-3as}+e^{-5as}-e^{-7as}+e^{-9as}-e^{-11as}+\ldots\right)$$

$$-\frac{1}{s}\left(e^{-2as}-e^{-4as}+e^{-6as}-e^{-8as}+e^{-10as}+\ldots\right)$$

$$f(t)=\left(u_a-u_{3a}+u_{5a}-\ldots\right)-\left(u_{2a}-u_{4a}+u_{6a-\ldots}\right)$$

## Exercise (3)

(1) Use the convolution theorem to find the following

(a) $L^{-1}\left(\dfrac{1}{(s-1)(s-2)}\right)$ (b) $L^{-1}\left(\dfrac{1}{s(s^2+1)}\right)$ (c) $L^{-1}\left(\dfrac{1}{s^2(s^2+1)}\right)$

(d) $L^{-1}\left(\dfrac{1}{s^2(s+4)^2}\right)$ (e) $L^{-1}\left(\dfrac{1}{(s^2+1)^3}\right)$

**Answer Question (1)**

(a) $L^{-1}\left(\dfrac{1}{(s-1)(s-2)}\right).=e^t*e^{2t}=\int_0^t e^{(t-x)}.e^{2x}dx=e^t\int_0^t e^x dx=e^t(e^t-1)$

(b) $L^{-1}\left(\dfrac{1}{s(s^2+1)}\right)=\int_0^t \sin(x)dx=-\cos(x)_0^t=1-\cos t$

(c) $L^{-1}\left(\frac{1}{s^2(s^2+1)}\right)=L^{-1}\left(\frac{1}{s^2}\cdot\frac{1}{(s^2+1)}\right)=t*\sin t=\int_0^t (t-x)\sin x dx$

$=\int_0^t t\sin x dx-\int_0^t x\sin x dx=\left[-t\cos x-(-x\cos x+\sin x)\right]_0^t$

$=t-\sin t$

(d) $L^{-1}\left(\frac{1}{s^2(s+4)^2}\right)=L^{-1}\left(\frac{1}{s^2}\cdot\frac{1}{(s+4)^2}\right)=t*te^{-4t}$

$=\int_0^x (t-x).xe^{-4x}dx=\int_0^x txe^{-4x}dx-\int_0^x x^2e^{-4x}dx$

(e) $L^{-1}\left(\frac{1}{(s^2+1)^3}\right)=L^{-1}\left(\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+1)^2}\right)$

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**Answer Question (2)**

(2) Use the convolution theorem to show that

(a) $\int_0^t \cos x\sin(t-x)dx=\frac{1}{2}t\sin t$

**Answer**

$L\left\{\int_0^t \cos x\sin(t-x)dx\right\}=L\{\cos x\}.L\{\sin x\}=\frac{s}{(s^2+1)^2}=-\frac{1}{2}\frac{d}{ds}\frac{1}{(s^2+1)}$

Then $\int_0^t \cos x\sin(t-x)dx=\frac{-1}{2}L^{-1}\left\{\frac{d}{ds}\frac{1}{(s^2+1)}\right\}=\frac{-1}{2}(-t)L^{-1}\left\{\frac{1}{(s^2+1)}\right\}=\frac{t}{2}\sin t$

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(b) $y(t)=\frac{1}{\omega}\int_0^t f(x)\sinh\omega(t-x)dx$ is a solution to the D.Eqn

$y''-\omega^2y=f(t),\ y(0)=y'(0)=0$

**Answer**

$$y(t)=\frac{1}{\omega}\int_0^t f(x)\sinh\omega(t-x)dx$$

$$s^2Y(s)-\omega^2Y(s)=F(s)$$

$$Y(s)=F(s)\frac{1}{s^2-\omega^2}=\frac{1}{\omega}F(s).L\{\sinh\omega t\}$$

$$y(t)=\frac{1}{\omega}\left(f(t)*\sinh t\right)=\frac{1}{\omega}\int_0^t f(x)\sinh(t-x)dx$$

Which is a solution for the differential Equation

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**<u>Answer Question (3)</u>**

(3) Solve the following integral Equations

(a) $y(t)=1+\int_0^t \cos x\ y(t-x)dx$ (b) $y(t)=\sin t+\int_0^t e^x\ y(t-x)dx$

(c) $y(t)=1+\int_0^t \sin x\ y(t-x)dx$ (d) $te^{-at}=\int_0^t y(x)\ y(t-x)dx$

(e) $y'(t)+\int_0^t y(t-x)dx=\cos t,\ \ y(0)=0$

**Answer (a)** (a) $y(t)=1+\int_0^t \cos(t-x)y(x)dx$

$$Y(s)=\frac{1}{s}+Y(s).\frac{s}{s^2+1}$$ then $$Y(s)=\frac{s^2+1}{s(s^2-s+1)}=\frac{A}{s}+\frac{Bs+C}{s^2-s+1}$$

$$Y(s)=\frac{s^2+1}{s(s^2-s+1)}=\frac{1}{s}+\frac{1}{s^2-s+1}$$ and $$y(t)=1-\frac{2}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t$$

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**Answer (b)** (b) $y(t) = \sin t + \int_0^t e^x \, y(t-x)dx$

$$Y = \frac{1}{s^2+1} + \frac{1}{s-1}Y \quad \Rightarrow Y\left(1 - \frac{1}{s-1}\right) = \frac{1}{s^2+1} \Rightarrow Y\left(\frac{s-2}{s-1}\right) = \frac{1}{s^2+1}$$

$$Y = \frac{s-1}{(s^2+1)(s-2)} = \frac{As+B}{s^2+1} + \frac{C}{s-2} = \frac{1}{5}\left(\frac{-s+3}{s^2+1} + \frac{1}{s-2}\right)$$

$$Y == \frac{1}{5}\left(\frac{-s}{s^2+1} + \frac{3}{s^2+1} + \frac{1}{s-2}\right) \text{ then } y(t) = \frac{1}{5}\left(-\cos t + 3\sin t + e^{2t}\right)$$

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**Answer (c)** (c) $y(t) = 1 + \int_0^t \sin x \; y(t-x)dx$

$$Y = \frac{1}{s} + \frac{1}{s^2+1}Y \quad \text{then} \quad \text{Y} = \frac{1}{s} + \frac{1}{s^3} \text{ and } y(t) = 1 + \frac{1}{2}t^2$$

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**Answer (d)** (d) $te^{-at} = \int_0^t y(x) \; y(t-x)dx$

Applly Laplace Transform for both sides we have

$$te^{-at} = \int_0^t y(x) \; y(t-x)dx$$

$$\frac{1}{(s+a)^2} = Y^2 \quad \Rightarrow Y = \pm\frac{1}{s+a} \text{ hence } y(t) = \pm e^{-at}$$

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**Answer (e)** (e) $y'(t) + \int_0^t y(t-x)dx = \cos t, \;\; y(0) = 0$

Applly Laplace Transform for both sides we have

$$sY - y(0) + \frac{1}{s}Y = \frac{s}{s^2+1} \qquad \Rightarrow \qquad (s^2+1)Y = \frac{s^2}{\left(s^2+1\right)}$$

$$Y = \frac{s^2}{\left(s^2+1\right)^2} \text{ hence } y(t) = \cos t * \cos t = \int_0^t \cos x \cos(t-x)dx$$

$$y(t) = \frac{1}{2}\int_0^t \left(\cos(2\mathrm{x}-\mathrm{t}) + \cos t\right)dx = \frac{1}{2}\left(\frac{1}{2}\sin(2\mathrm{x}-\mathrm{t}) + x\cos t\right)_0^t$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin t + t\cos t - \frac{1}{2}\sin t\right) = \frac{1}{2}t\cos t$$

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**<u>Answer Question (4)</u>**

(4) Solve the following initial value problems by using Laplace transform

(a) $y' - y = \cos t, \qquad y(0) = 1$

(b) $y' + y = t^2 \mathrm{e}^t, \qquad y(0) = 2$

(c) $y'' + 4y = \sin t, \qquad y(0) = 1,\ y'(0) = 0$

(d) $y'' - 2y' - 3y = te^t, \qquad y(0) = 2,\ y'(0) = 1$

(e) $y'' + y' = f(t), \qquad y(0) = 1,\ y'(0) = -1$

**Answer (a)** (a) $y' - y = \cos t, \qquad y(0) = 1$

$$sY(s) - y(0) - Y(s) = \frac{1}{s^2+1}$$

$$sY(s) - 1 - Y(s) = \frac{1}{s^2+1} \quad \Rightarrow \quad (s-1)Y(s) = \frac{1}{s^2+1} + 1$$

$$Y(s) = \frac{1}{(s-1)(s^2+1)} + \frac{1}{s-1}$$

$$y(t) = e^t * \sin t + e^t = \int_0^t e^{(t-x)} \sin x dx + e^t = \frac{-1}{2}\left(\cos t + \sin t\right) + \frac{1}{2}e^t + e^t$$

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**Answer (b)** (b) $y' + y = t^2 e^t, \qquad y(0) = 2$

$$sY(s) - y(0) + Y(s) = \frac{2}{(s-1)^3}$$

$$sY(s) - 2 + Y(s) = \frac{2}{(s-1)^3} \Rightarrow (s+1)Y(s) = \frac{2}{(s-1)^3} + 2$$

$$Y(s) = \frac{2}{(s+1)(s-1)^3} + \frac{2}{s+1} \text{Hence } y(t) = \frac{1}{2}e^t\left(t^2 - t + \frac{1}{2}\right) + 2e^{-t}$$

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**Answer (c)** (c) $y'' + 4y = \sin t, \qquad y(0) = 1, \; y'(0) = 0$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s^2+1}$$

$$s^2Y(s) - s + 4Y(s) = \frac{1}{s^2+1} \Rightarrow s^2Y(s) + 4Y(s) = \frac{1}{s^2+1} + s$$

$$Y(s) = \frac{1}{(s^2+4)(s^2+1)} + \frac{s}{(s^2+4)}$$

**For** $\frac{1}{(s^2+4)(s^2+1)} = \frac{Ax+B}{(s^2+1)} + \frac{Cx+D}{(s^2+4)} = \frac{1}{3}\left[\frac{1}{(s^2+1)} - \frac{1}{(s^2+4)}\right]$

**Then** $Y(s) = \frac{1}{3}\left[\frac{1}{(s^2+1)} - \frac{1}{(s^2+4)}\right] + \frac{s}{(s^2+4)}$

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + \cos 2t$$

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**Answer (d)** (d) $y'' - 2y' - 3y = te^t, \quad y(0) = 2, \; y'(0) = 1$

$$s^2Y(s) - sy(0) - y'(0) - 2\left[sY - y(0)\right] - 3Y(s) = \frac{1}{s-1}$$

$$s^2Y(s) - 2s - 1 - 2\left[sY - 2\right] - 3Y(s) = \frac{1}{s-1}$$

$$s^2Y(s) - 2sY - 3Y(s) = \frac{1}{s-1} + 2s - 3$$

$$Y(s)=\frac{1}{(s^2-2s-3)(s-1)}+\frac{2s-3}{(s^2-2s-3)}=\frac{2s^2-5s+4}{(s-3)(s+1)(s-1)}$$

$$=\frac{7/8}{(s-3)}+\frac{11/8}{(s+1)}+\frac{-1/4}{(s-1)}$$

$$y(t)=\frac{1}{8}\left(7e^{3t}+11e^{-t}-2e^{t}\right)$$

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**Answer (e)** (e) $y''+y'=f(t), \qquad y(0)=1,\ y'(0)=-1$

$$s^2Y(s)-sy(0)-y'(0)+\left[sY-y(0)\right]=F(s)$$

$$s^2Y(s)-s+sY=F(s)$$

$$(s^2+s)Y(s)=F(s)+s$$

$$Y(s)=\frac{F(s)+s}{(s^2+s)}=\frac{F(s)}{s(s+1)}+\frac{1}{(s+1)}$$